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ON GENERALIZED LEFT DERIVATIONS AS HOMOMORPHISONS AND ANTI - HOMOMORPHISMS

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Abstract

Let R be a 2-torsion free prime ring and $I \neq 0$ an Ideal of R and F is a generalized left derivation of R. In this paper we study the following conditions such that either $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ for all , $xy \in I$, then R is commutative.

1. Introduction

Throughout the present paper R will denote an associative ring with center $Z(R)$. For any $x, y \in R$ the symbol $[x, y]$ stands for the commutator $xy - yx$ the symbol (xoy) stands for the anti commutator $xy + yx$, thus the commutator identities.

(i) $[xy, z] = x[y, z] + [x, z]y$

(ii) $[x, yz] = y[x, z] + [x, y]z$ for all $x, y, z \in R$.

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Recall that a ring R is called prime if for any $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$ however, in case prime rings Bell and Kappe [1] established that if I is a non zero right ideal of a prime ring R and $d : R \to R$ is a derivation of R such that d acts as a homomorphism or as anti-homomorphism on I. Then $d = 0$ on R. In the year 1990, Bresar and vukman [2] introduced the notion of a left derivation as follows. An additive mapping $d : R \to R$ is said to be a left derivation if $d(xy) = xd(y) + yd(x)$ holds for all $x, y \in R$. In [3], Bresar was introduced the generalized derivation. An additive mapping $F : R \to R$ is said to be generalized left derivation on R if there exists a left derivation $d : R \to R$ such that $F(xy) = xF(y) + yd(x)$ holds for all $x, y \in R$. Nadeem-Ur-Rehman [4] proved that Let R be a 2-torsion free prime ring and I be a non zero ideal of R. Suppose $F : R \to$R is a non zero generalized derivation with d.

(i) If F acts as a homomorphism on I and if $d \neq 0$, then R is commutative.

(ii) If F acts as anti-homomorphism on I and if $d \neq 0$, then R is commutative.

In the present paper our objective is to extend this result for generalized left derivation acting on ideals in prime ring.

2. Preliminaries

Lemma 2.1: [5,lemma 3] : Let R be a prime ring and I be a non zero right ideal of R. If d is a non zero derivation on R then d is non zero on I.

Lemma 2.2 [6, lemma 2] : If a prime ring R contains a non zero commutative righ ideal, then R is commutative.

3. Main Results

Theorem 3.1 : Let R be a 2-torsion free prime ring and I be a non zero Ideal of R. Suppose $F : R \to R$ is a non zero generalized left derivation with left derivation d where d commutes with I

(i)If $F(xy) = F(x)\ F(y)$ for all $x, y \in I$ and if $d \neq 0$ (That is if F acts as a homomorphism on I) and if $d \neq 0$ then R is commutative.

(ii) If $F(xy) = F(y)F(x)$ for all $x, y \in I$ and if $d \neq 0$ (That is if F acts as an anti homomorphism on I) and if $d \neq 0$ then R is commutative.

Proof : (i) If F acts as a homomorphism on I then we have, for all $xy \in I$.

$$F(xy) = xF(y) + yd(x) = F(x)F(y) \tag{3.1}$$

Replace x by zx, Since F is a generalized left derivation

$$F(zxy) = zF(xy) + xyd(z)\text{“ for all } x, y, z \in I \tag{3.2}$$

$$F(zxy) = F(zx)F(y) \text{ for all } x, y, z \in I \tag{3.3}$$

$$\begin{aligned} F(zxy) &= F(zx)F(y) \text{ for all } x, y, z \in I \\ &= \{zF(x) + xd(z)\}F(y) \text{ for all } x, y, z \in I \\ &= zF(x)F(y) + xd(z)F(y) \text{ for all } x, y, z \in I. \end{aligned}$$

Using (3.1)

$$F(zxy) = zF(xy) + xd(z)F(y) \text{ for all } x, y, z \in I \tag{3.4}$$

Comparing (3.2) and (3.4)

$$zF(xy) + xyd(z) = zF(xy) + xd(z)F(y) \text{ for all } x, y, z \in I.$$

We get $xyd(z) = xd(z)F(y)$ for all $x, y, z \in I$

$$xyd(z) + xd(z)y - xd(z)y = xd(z)F(y) \text{ for all } x, y, z \in I$$

$$xyd(z) - xd(z)y - xd(z)F(y) - xd(z)y \text{ for all } x, y, z \in I.$$

$$X[y, d(z)] = xd(z)(F(y) - y) \text{ for all } x, y, z \in I.$$

d commutes with I we get

$$xd(z)(F(y) - y) = 0 \text{ for all } x, yz \in I \tag{3.5}$$

$$d(z)x(F(y) - y) = 0 \text{ for all } x, y, z \in I.$$

Thus, primness of Rforces that Either $d(z) = 0$ or $(F(y) - y) = 0$ for all $x, y, z \in I$. If $d(z) = 0$, for all $z \in I$ then $d = 0$.
A contradiction.

On the other hand if $F(y) - y = 0$

$$\Rightarrow F(y) = y \text{ for all } y \in I$$
$$\Rightarrow F(x) = x \text{ for all } x \in I.$$

Now

$$\begin{aligned} xy &= F(x)F(y) \quad \text{(since } F \text{ is homomorphism)} \\ &= F(xy) \quad \text{(since } F \text{ is generalized left derivation)} \end{aligned}$$

$$xy = xF(y) + yd(x) \text{ for all } x, y \in I$$

$$xy - xF(y) = yd(x) \text{ for all } x, y, z \in I$$

$$x(y - F(y)) = yd(x) \text{ for all } x, y, z \in I$$

$$(y - y) = yd(x) \text{ for all } x, y, z \in I$$

We get

$$yd(x) = 0 \text{ for all } x, y \in I \tag{3.6}$$

$$Id(x) = 0 \text{ for all } x \in I$$

Again since $i \neq 0$ and R is prime we get $d(x) = 0$ for all $x \in I$ and hence $d = 0$. Again contradiction.

(ii) If F acts as an anti homomorphism on I and if $d \neq 0$ then R is commutative.

Proof : If F acts as an ant homomorphism on

$$I.F(xy) = F(y)F(x) \tag{3.7}$$

Since F is generalized left derivation

$$F(xy) = xF(y) + yd(x) = F(y)F(x) \tag{3.8}$$

Replace y by xy.

$$F(xy)F(x) = F(xxy) = xF(xy) + xyd(x) \tag{3.9}$$

Since F is an anti hommorphism

$$F(xxy) = F(xy)F(x) \tag{3.10}$$

right multiplying (3.8) with $F(x)$

$$\begin{aligned}(xy)F(x) &= \{xF(y) + yd(x)\}F(x) \\ &= xF(y)F(x) + yd(x)F(x)\end{aligned}$$

$$F(xy)F(x) = xF(xy) + yd(x)F(x) \tag{3.11}$$

Equating (3.9) and (3.11)

$$xF(xy) + xyd(x) = xF(xy) + yd(x)F(x)$$

we get

$$xyd(x) = yd(x)F(x) \tag{3.12}$$

replace y by zy in (3.12)

$$xzyd(x) = zyd(x)F(x) \tag{3.13}$$

Left multiply (3.12) with z we get

$$zxyd(x) = zyd(x)F(x) \text{ for all } x, y, z \in I \tag{3.14}$$

Equating (3.13) and (3.14)

$$xzyd(x) = zxyd(x) \text{ for all } x, y, z \in I$$

$$[x, z]yd(x) = 0 \text{ for all } x, y, z \in I \tag{3.15}$$

$$\Rightarrow [x, z]Idx = 0 \text{ for all } x, y, z \in I.$$

Thus primness of R implies that for each $x, z \in I$ either $[x, z] = 0$ or $d(x) = 0$. If $d(x) = 0$ then $d = 0$.

If $[x, z] = 0$ for all $x, y, z \in I$, by lemma 2.2 R is commutative.

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